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Existence of global attractor in strongly continuous semigroups $\{T_t\}$ that has Lyapunov function in a metric space

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ABSTRACT

Many Problems in our life can be formed by mathematics modelling to simplify its solutions. Most of the problems is formed in differential equation form. And the one interesting part of mathematics to be studied is problems in infinite dimensional spaces. `Global attractor' is a basic concept and tool to study asymptotic behaviors of solutions of nonlinear evolution equations. A global attractor is an invariant compact set absorbing all of the bounded sets as time goes to infinity and by use of this method we can reduce problems for motions in infinite dimensional spaces to some finite dimensional ones. In metric space there is parameter family of maps that called C^0 semigroup (strongly continuous semigroups) and we will discuss the existence of global attractor in it. specially for global attractor in strongly continuous semigroups $\{T_t\}$ that has Lyapunov function.

Keywords : *Global Attractor, strongly continuous semigroups, Lyapunov function*.

1. Introduction

Global attractor is a method to describe asymptotic behaviors of solutions of nonlinear evolution equations. By global attractor method we can reduce problems for motions in infinite dimensional space to some finite dimensional ones.

Strongly continuous semigroups is a parameter family $\{T_t\}$ of maps $T_t : V \to V, t \geq 0$. Where "t" is notation of time that goes to infinity time. By use global attractor method we can solve some problems nonlinear evolution equations easily on strongly continuous semigroups. One interesting space for the authors is metric space. By properties of strongly continuous semigroups $\{T_t\}$ and properties of Lyapunov function we will show the existence of a global attractor.

In this paper we will discuss the existence of global attractor in strongly continuous semigroups $\{T_t\}$ that has Lyapunov function of a metric space with additional assumption for strongly continuous semigroups $\{T_t\}$.

2. Existence of global attractor in strongly continuous semigroups $\{T_t\}$ that has Lyapunov function of a metric space

In 2000, Cholewa, Jan W.,Tomasz Dlotko. [1] define strongly continuous semigroups, asymptotically smooth and Lyapuov function as follows.

Definition 1. Let V be a matric space. A one parameter family $\{T_t\}$ of maps $T_t : V \to V, t \geq 0$, is called a C^0 semigroup (strongly continuous semigroups) if

(i) T_0 is the identity map on V,
(ii) $T_{t+s} = T_t \circ T_s$ for all $t, s \geq 0$,
(iii) The function
$$[0, \infty) \times V \ni (t, x) \to T_t(x) \in V.$$
Is continuous at each point $(t, x) \in [0, \infty) \times V$.

Definition 2. Let $\{T_t\}$ be a C^0 semigroup in a metric space V. A function $\mathcal{L} : V \to R$ is said to be a Lyapunov function for $\{T_t\}$ if

(i) $\mathcal{L}$ is continuous on V and bounded below,
(ii) For each $v \in V$ function $(0, +\infty) \ni t \to \mathcal{L}\,(T_t(v)) \in R$ is nonincreasing,
(iii) For any $v \in V$ we have
$\mathcal{L}\,(T_t(v)) = const$. For all $t \geq 0$ implies that $T_t(v) = v$ for all $t \geq 0$.

Definition 3. Semigroups- $\{T_t\}$ is asymptotically smooth if and only if for every nonempty subset, closed, bounded and $\{T_t\} - invariant\ set\ W \subset V$ contains compact subset C that attracts W.

3. Result and Discuss

In this section it should prove that $C^0\ semigroup\ \{T_t\}$ that has Lyapunov function admits Global Attractor under an additional regularity assumption on $\{T_t\}$.
Under the following assumptions,
Assumption A. $\{T_t\}$ is an asymptotically smooth $C^0\ semigroup$ on metric space V.
Assumption B. $\{T_t\}$ satisfies the condition :
$$\forall_{\substack{B \subset V \\ B\ bounded}} \exists_{t_B \geq 0} \bigcup_{t \geq t_B} T_t(B)\ is\ bounded\ in\ V.$$
Assumption C. $A: \{v \in V | \forall_{t \geq 0}, T_t(v) = v\} \subset B\ bounded\ subset\ of\ V.$

Our result is:
Main Theorem. Under the **assumption A**, **B** and **C**. if $\{T_t\}$ has Lyapunov function $\mathcal{L}$, then $\{T_t\}$ has a global attractor.

To prove the theorem, we need to use some theorems in real analysis and functional analysis.
Some theorems from [2] Kreyszig, Erwin and [3] Bartle, Robert G., Donald D. Sherbert.will be used to prove the main theorem.

Theorem 1. Let X, Y are nonempty subset and $\mathbb{N}$ is set of natural number. If $f: X \longrightarrow Y$ is a mapping from X onto Y and if the set $\{X_i | i \in \mathbb{N}\}$ is any subset of X, then

$$f\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \bigcup_{i\in\mathbb{N}} f(X_i)$$

And

$$f\left(\bigcap_{i\in\mathbb{N}} X_i\right) = \bigcap_{\in\mathbb{N}} f(X_i).$$

Theorem 2. A compact set is absolutely closed.

Theorem 3. Let V is a metric space, K is compact $\subset V$. If $F \subset K$ and F is closed, then F is compact.

Theorem 4.
Let $\{T_t\}$ be a C^0 semigroup on a metric space V. If $\{T_t\}$ is asymptotically smooth, if B is a nonempty subset of V, and for some number $t_B \geq 0$, the set

$$\bigcup_{s\geq t_B} T_s(B)$$

Is bounded, then $\omega(B)$ is nonempty, compact and invariant. Furthermore $\omega(B)$ attracts B.

Proof of main theorem.
The existence of global attractor in strongly continuous semigroup which has Lyapunov function will be explained below.
Step I
To prove the theorem we start to show the existence bonded set $\mathcal{O} \subset V$ such that for each compact set $C \subset V$ has neighborhood $\mathcal{N}_C$ absorbed by $\mathcal{O}$.
By **Assumption B** we get that for each point $v \in V$ the corresponding $\omega - limit\ set\ \omega(v)$ lies within the bounded set $\mathcal{E}$ of all equilibria for $\{T_t\}$:

$$\mathcal{E} := \{w \in V: T_t(w) = w\ for\ all\ t \geq 0\}.$$

Let $\mathcal{L}$ be a Lyapunov function for $\{T_t\}$. Based on **Theorem 3** our assumption ensures that the $\omega - limit\ set\ \omega(v)$ of any element $v \in V$ is nonempty and attracts v. For fixed $v \in V$, let us choose any $y \in \omega(v)$ and a sequence $t_n \to +\infty$, such that

$$\lim_{n\to\infty} T_{t_n}(v) = y.$$

Since $\mathcal{L}$ is bounded below we have, for some $\beta \in \mathbb{R}$,

$$\inf_{t\geq 1} \mathcal{L}(T_t(v) = \beta.$$

From the continuity of $\mathcal{L}$ and the semigroup properties of $\{T_t\}$ we also obtain

$$\mathcal{L}(T_t(y)) = \mathcal{L}\left(T_t\left(\lim_{n\to\infty} T_{t_n}(v)\right)\right) = \lim_{n\to\infty} \mathcal{L}\left(T_t\left(T_{t_n}(v)\right)\right)$$

$$= \lim_{n\to\infty} \mathcal{L}(T_{t+t_n}(v)) = \beta \; for \; each \; t \geq 0.$$

This proves that y is an equilibrium point and therefore the union $\bigcup_{v\in V} \omega(v)$ is contained in $\mathcal{E}$. Recalling that $\bigcup_{v\in V} \omega(v)$ attracts points of V.
Using the above condition there exist nonempty, bounded set $\bigcup_{v\in V} \omega(v) := W_0 \subset V$ attracts each point of V. Let $\mathcal{N}_{W_0}$ be any bounded open neighborhood of W_0 Therefore

$$\forall_{v\in V} \exists_{t_0\geq 0} \forall_{t\geq t_0} T_t(v) \subset \mathcal{N}_{W_0}.$$

So $\mathcal{N}_{W_0}$ absorbs points in V.
from the continuity of $\{T_t\}$ we get and the fact that $\mathcal{N}_{W_0}$ absorbs points in V, we get

$$\forall_{v\in V} \forall_{\varepsilon>0} \exists_{\delta\geq 0} T_t(B_V(v,\delta)) \subset B_V(T_t(v),\varepsilon).$$

Since $\mathcal{N}_{W_0}$ absorb points of V, thus for $B_V(T_t(v),\varepsilon) \subset \mathcal{N}_{W_0}$, we choose $B_V(v,\delta)$ such that $T_t(B_V(v,\delta)) \subset \mathcal{N}_{W_0}$, so

$$\forall_{v\in V} \exists_{B_V(v,\delta)} \exists_{t_0\geq 0} T_t(B_V(v,\delta)) \subset \mathcal{N}_{W_0}.$$

Let we choose $t_{\mathcal{N}_{W_0}} \geq 0$, such that

$$\mathcal{O} := \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0})$$

Bounded.
By **assumption A**, $\{T_{t'}\}$ strongly continuous semigroup, then $\mathcal{O}$ is positive invariant

$$T_{t'}(\mathcal{O}) \subset \mathcal{O}, \qquad t' \geq 0.$$

It is shown

$$T_{t'}(\mathcal{O}) = T_{t'}\left(\bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0})\right), \; t' \geq 0$$

By **Theorem 4**

(5) $$T_{t'}(\mathcal{O}) = T(t')\left(\bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0})\right) = \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'}\left(T_t(\mathcal{N}_{W_0})\right)$$

Since $\{T_{t'}\}$ and $\{T_t\}$ are $C^0 \; Semigroup$, we have

$$\bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'}\left(T_t(\mathcal{N}_{W_0})\right) = \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'+t}(\mathcal{N}_{W_0})$$

Next we have

(6) $$\cup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'+t}(\mathcal{N}_{W_0}) = T_{t'+t_{\mathcal{N}_{W_0}}}(\mathcal{N}_{W_0}) \cup T_{t'+t_{\mathcal{N}_{W_0}}+1}(\mathcal{N}_{W_0}) \cup \ldots$$

Since $t' \geq 0$,

$$\bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0}) = T_{t_{\mathcal{N}_{W_0}}}(\mathcal{N}_{W_0}) \cup T_{t_{\mathcal{N}_{W_0}}+1}(\mathcal{N}_{W_0}) \cup \ldots \cup T_{t'+t_{\mathcal{N}_{W_0}}}(\mathcal{N}_{W_0}) \cup T_{t'+t_{\mathcal{N}_{W_0}}+1}(\mathcal{N}_{W_0}) \cup \ldots$$

(7) $$\cup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0}) = T_{t_{\mathcal{N}_{W_0}}}(\mathcal{N}_{W_0}) \cup T_{t_{\mathcal{N}_{W_0}}+1}(\mathcal{N}_{W_0}) \cup \ldots \cup \left\{\cup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'+t}(\mathcal{N}_{W_0})\right\} \cup \ldots$$

Therefore, by equation **(5)**, **(6)** and **(7)**. We have

$$T_{t'}(\mathcal{O}) = \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_{t'+t}(\mathcal{N}_{W_0}) \subset \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0}) = \mathcal{O}$$

So, we have $T_{t'}(\mathcal{O}) \subset \mathcal{O}$. we say $\mathcal{O}$ is positive invariant.
Next, since $\mathcal{N}_{W_0}$ absorbs each point in V.

$$\forall_{v\in V}\exists_{t_0\geq 0} \; such\ that\ T_t(v) \subset \mathcal{N}_{W_0}, \qquad t \geq t_0$$

Thus there exist $t_0 \geq 0$ such that

$$T_{t_{\mathcal{N}_{W_0}}}(T_t(v)) \subset T_{t_{\mathcal{N}_{W_0}}}(\mathcal{N}_{W_0}) \subset \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0}) = \mathcal{O}, \qquad t_{\mathcal{N}_{W_0}} \geq t_0$$

Then we say $\mathcal{O}$ absorbs each point in V.
By assumption $\{T(t)\}$ is strongly continuous semigroup, we have

$$T_{t_{\mathcal{N}_{W_0}}}(T_t(v)) = \left(T_{t+t_{\mathcal{N}_{W_0}}}(v)\right)$$

Based on and $\mathcal{O}$ is absorb point in V,

(8) $$\forall_{v\in V}\exists_{t_v=t_{\mathcal{N}_{W_0}}+t\geq 0}\,\exists_{B_V(v,\delta)} T_t(B_V(v,\delta)) \subset \mathcal{O},\ t \geq t_v$$

Let given any compact set $C \subset V$, thus by the compactness of C we have

$$C \subset \bigcup_{i=1}^{n} B_V(v_i,\delta_i) = \mathcal{N}_C$$

By properties of operator $T(t)$, thus

$$T_t(C) \subset T_t(\mathcal{N}_C) = T_t\left(\bigcup_{i=1}^{n} B_V(v_i,\delta_i)\right) = \bigcup_{i=1}^{n} T_t(B_V(v_i,\delta_i))$$

Therefore, by equation **(8)**. We have

(9) $$T_t\big(B_V(v_i,\delta_i)\big) \subset \mathcal{O},\ t \geq t_{v_i}$$

For , $t \geq \max\{,\ t_{v_1}, t_{v_2}, \dots, t_{v_n}\}$ we have $T_t(\mathcal{N}_C) \subset \mathcal{O}$.
Thus, there exist bounded set $\mathcal{O} \subset V$ such that each compact set $C \subset V$ have neighborhood $\mathcal{N}_C$ is absorbed by $\mathcal{O}$.

Step II

Now we will show there exist nonempty set $\mathcal{A} \subset V$, compact and $\{T_t\} - invariant$ attracts each bounded subset of V.
Let $B \subset V$ is nonempty bounded subset. By **assumption A** and **assumption B** according to **Theorem 3** we have $\omega(B)$ is compact and attracts B. In other words :

$$\forall_{\mathcal{N}_{\omega(B)}} \exists_{t_B \geq 0}\, T_{t_1}(\mathrm{B}) \subset \mathcal{N}_{\omega(B)}, \qquad t_1 \geq t_B.$$

Based on **(9)** there exist some neighborhood of compact set $\omega(B)$. and we note $\mathcal{N}_{\omega(B)}$ which absorbed by $\mathcal{O}$,

$$\exists_{t_0 \geq 0}\, T_{t_3}\big(\mathcal{N}_{\omega(B)}\big) \subset \mathcal{O}, \qquad t_2 \geq t_0$$

We set $t_3 = t_2 + t_1$,

$$T_{t_3}(B) = T_{t_2+t_1}(B)$$
$$= T_{t_2}\Big(T_{t_1}(B)\Big) \subset T_{t_2}\big(\mathcal{N}_{\omega(B)}\big) \subset \mathcal{O}.$$

Thus, there exist $\tau_B = t_B + t_0$ such that

(10) $$\forall_{B \subset V} \exists_{\tau_B \geq 0}\, T_{t_3}(B) \subset \mathcal{O},\ t_3 \geq \tau_B.$$

Let $\mathcal{A} := \omega(\mathcal{O})$, based on assumption A and Theorem 3. We have $\mathcal{A} := \omega(\mathcal{O})$ nonempty, compact and invariant. Furthermore $\mathcal{A}$ attracts $\mathcal{O}$.
We will show $\mathcal{A}$ attracts each bounded subset of attract V.
$\mathcal{A}$ attracts $\mathcal{O}$. It is mean

$$\forall_{\mathcal{N}_{\mathcal{A}}} \exists_{t_0 \geq 0}\, T_t(\mathcal{O}) \subset \mathcal{N}_{\mathcal{A}}, \qquad t \geq t_0.$$

By condition (9),
We set $t_4 = t + t_2$,

$$T_{t_4}(B) = T_{t+t_2}(B)$$
$$T_t\Big(T_{t_2}(B)\Big) \subset T_t(\mathcal{O}) \subset \mathcal{N}_{\mathcal{A}}$$

Thus, there exist $\tau_2 = t_C + \tau_B$ such that

$$\forall_{B \subset V} \exists_{\tau_2 = t_C + \tau_B \geq 0}\, T_{t_4}(B) \subset \mathcal{N}_{\mathcal{A}}, \qquad t_4 \geq \tau_2.$$

Therefore $\mathcal{A}$ attracts each bounded subset of V.
Then there exist global attractor which is nonempty, invariant, compact and attracts each bounded subset of V.■

4. Conclusion

In conclusion of this paper we would like to remark that under additional regularity assumption for $\{T_t\}$ that has Lyapunov function in metric space such that $\{T_t\}$ has global attractor. The additional regularity assumption is $\{T_t\}$ is asymptotically smooth, Orbit of bounded set in V is bounded and all the equilibria of $\{T_t\}$ lie in bounded subset of V.

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